Total No. of Questions : 8]

SEAT No. :

P1901

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[4921]-405
M.A/M.Sc.
MATHEMATICS
MT - 805 : Lattice Theory
(2008 Pattern) (Semester-IV)

Time : 3 Hours] ***[Max. Marks : 80***

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to the right indicate full marks.*

Q1) a) Let $\mathbb{N}_0$ be the set of all non-negative integers. Define $m \leq n$ if there exists $k \in \mathbb{N}_0$ such that $n = km$. Prove that $\mathbb{N}_0$ is a lattice under this relation. **[5]**

b) Let I be an ideal and let D be a dual ideal. If $I \cap D \neq \emptyset$ then show that $I \cap D$ is a convex sublattice and every convex sublattice can be expressed in this form in one and only one way. **[5]**

c) Prove that Con(L), the set of all congruence relations of a lattice L, is a lattice. **[6]**

Q2) a) Prove that a lattice L can be embedded in the ideal lattice $Id(L)$ and this embedding is onto if L is finite. **[6]**

b) Show that the chain with five elements $C_5 \cong L \times K$ then L or K has only one element. **[2]**

c) Prove that the lattice L is modular if and only if it does not contain a sublattice isomorphic to N_5. **[8]**

Q3) a) Let L be a pseudocomplemented lattice, $S(L) = \{a^* | a \in L\}$. Then prove that the partial ordering of L parially orders $S(L)$ and makes $S(L)$ into a Boolean lattice. For $a,b \in S(L)$, we have $a \wedge b \in S(L)$ and the join in $S(L)$ is described by $a \vee b = (a^* \wedge b^*)^*$ **[8]**

b) Define homomorphism of lattices and prove that a homomorphic image of a lattice L is isomorphic to a suitable quotient lattice of L. **[8]**

P.T.O.

Q4) a) Show that the following inequalities hold in any lattice **[4]**

i) $(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z)$;

ii) $(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee (x \wedge z))$.

b) Let L and K be lattices, let θ be a congruence relation of L, and let ϕ be a congruence relation of K. Define the relation $\theta \times \phi$ on $L \times K$ by $< a,b > \equiv < c,d > (\theta \times \phi)$ if and only if $a \equiv c(\theta)$ and $b \equiv d(\phi)$. Then show that $\theta \times \phi$ is a congruence relation on $L \times K$ and conversely, every congruence relation of $L \times K$ is of this form. **[6]**

c) Prove that in a distributive lattice L, if the ideals $I \vee J$ and $I \wedge J$ are principal then so are I and J. **[6]**

Q5) a) Let L be a distributive lattice with 0. Show that $a^{\perp} = \{x \in L \mid x \wedge a = 0\}$ is an ideal of L. **[4]**

b) Prove that a maximal ideal of a distributive lattice is prime but not conversely. **[4]**

c) State and prove Nachbin theorem. **[8]**

Q6) a) Prove that every finite distributive lattice is isomorphic to ring of sets. **[8]**

b) Let L be a distributive lattice, let I be an ideal, let D be a dual ideal of L, and let $I \cap D = \emptyset$. Then prove that there exists a prime ideal P of L such that $P \supseteq I$ and $P \cap D = \emptyset$. **[8]**

Q7) a) State and prove Jordan-Hölder Theorem for semimodular lattices. **[7]**

b) Prove that every modular lattice satisfies the upper covering conditions. **[5]**

c) Prove that the dual of a distributive lattice is distributive. **[4]**

Q8) a) Prove that every prime ideal is a meet-irreducible element of a ideal lattice but not conversely. **[6]**

b) Prove that a lattice L is distributive if and only if for any two ideals I, J of L, $I \vee J = \{i \vee j \mid i \in I, j \in J\}$. **[5]**

c) Let $<P;\leq>$ be a poset in which inf H exists for all $H \subseteq P$. Show that $< P;\leq >$ is a lattice. **[5]**

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