Total No. of Questions :8]

SEAT No. :

P1893

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[4921] - 302
M.A. / M.Sc.
MATHEMATICS
MT - 702 : Ring Theory
(2008 Pattern) (Semester -III) (Old Course) (Non Credit Course)

Time :3 Hours] *[Max. Marks :80*

Instructions to the candidates:

1) ***Attempt any five questions.***
2) ***Figures to right indicate full marks.***

Q1) a) Prove that any subring of a field which contains the identity is an integral domain. Is the converse true? **[6]**

b) If $a \in z$ is an integer then prove that the element $\overline{a} \in \frac{z}{nz}$ is nilpotent if and only if every prime divisor of n is also a divisor of a. In particular determine the nilpotent elements of the ring $\frac{\mathrm{Z}}{12\mathrm{Z}}$. **[6]**

c) Prove that each Boolean ring is commutative. **[4]**

Q2) a) If $\phi: \mathrm{R} \to \mathrm{S}$ is a ring homomorphism then show that

i) Kernel of ϕ is an ideal of R

ii) Image of ϕ is a subring of S

iii) $\frac{\mathrm{R}}{kcer\phi} \simeq \phi(\mathrm{R})$. **[6]**

b) Prove that the ring 2z and 3z are not isomorphic. **[6]**

c) Find all homomorphic images of Z. **[4]**

P.T.O.

Q3) a) If R is a commutative ring with unity 1. Then prove that an ideal M of R is maximal ideal if and only if the quotient ring $\frac{R}{M}$ is a field. **[8]**

b) Show that the ideal I = (2, x) generated by 2 and x in Z[x] is not a principal ideal. **[4]**

c) If $P(x) = x^2 + x + 1$ is an element of the polynomial ring $R = Z_2[x]$ then prove that the quotient ring $\overline{R} = \frac{Z_2[x]}{(x^2+x+1)}$ has 4 elements and additive group of $\overline{R}$ is isomorphic to Klein 4 - group. **[4]**

Q4) a) If A_1 and A_2 are comaximal ideals in the ring R then prove that

$$\frac{R}{A_1A_2} = \frac{R}{A_1 \cap A_2} \simeq \frac{R}{A_1} \times \frac{R}{A_2}.$$ **[10]**

b) Prove or disprove **[6]**

i) If R is a field then its field of fractions is also R.

ii) If R = 2Z in the ring of even integers then its field of fraction is Q, the set of retionals.

iii) If R and S are non - zero rings then R×S can be a field.

Q5) a) Prove that every ideal in a Euclidean domain is principal. **[8]**

b) Using above result (a), show that the ring $Z[\sqrt{-5}]$ is not Euclidean domain. **[6]**

c) Prove that the quotient ring $\frac{Z(i)}{(1+i)}$ is a finite ring and find its all elements. **[2]**

Q6) a) Prove that in a PID a non - zero non unit element is prime if and only if it is an irreducible. **[8]**

b) If a = 3 + 2i and b = 2 – 3i are two elements in Z[i] then find q and r in Z[i] such that $a = bq + r$ and N (r) < N (b) where N ($x + iy$) = $x^2 + y^2$. **[4]**

c) If the prime number p is either 2 or is an odd prime congruent to 1 modulo 4 then prove that p divides an integer of the form $n^2 + 1$. **[4]**

Q7) a) If R is a PID then prove that there exist a multiplicative Dedekind - Hasse norm on R. **[8]**

b) If F is a field then prove that the set R of polynomials in F(x) whose coefficient of x is equal to zero is a subring of F[x] and that R is not UFD. **[6]**

c) Prove or disprove. **[2]**

The ring Z[2i] is a UFD.

Q8) a) Find all monic irreducible polynomials of degree ≤ 2. **[4]**

b) Let I be a proper ideal in the integral domain R and let P(x) be a nonconstant monic polynomial in R[x]. If the image of P(x) in $\frac{R}{I}[x]$ cannot be factored in $\frac{R}{I}[x]$ into two polynomials of smaller degrees then show that P(x) is irreducible in R[x]. **[8]**

c) Show that the polynomial $x^3 + x + 1$ is irreducible in Z [n]. **[4]**