

Total No. of Questions : 8]

SEAT No. :

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[5121]-302

M.A./M.Sc. (Semester - III)

MATHEMATICS

MT-702 : Field Theory

(2013 Pattern) (Credit System)

Time : 3 Hours]

[Max. Marks : 50

Instructions to the candidates:

- 1) *Attempt any five questions.*
- 2) *Figures to the right indicate full marks.*

Q1) a) Let α be an algebraic over a field F and $F(\alpha)$ be the field generated by α over F then prove that, $F(\alpha) \cong F[x]/\langle m_\alpha(x) \rangle$ hence show that $[F(\alpha); F] = \deg(m_\alpha(x))$ where $M_\alpha(x)$ is minimal polynomial for α over F .

[5]

b) Show that $P(x) = x^2 + 1$ is an irreducible polynomial over the field Z_3 . Find an extension K of Z_3 in which $P(x)$ has a root.

[3]

c) Show that the characteristic of a field F is either zero or a prime.

[2]

Q2) a) Prove that the extension K/F is finite if and only if K is generated by a finite number of algebraic elements over F .

[5]

b) Show that $[Q(\sqrt[6]{2}):Q] = 6$ and hence show that $x^3 - \sqrt{3}$ is irreducible polynomial over $Q(\sqrt{2})$.

[3]

c) Determine the degree of $\alpha = 2 + \sqrt{3}$ over Q .

[2]

Q3) a) Find the splitting field of $f(x) = x^4 - 2 \in Q[x]$ over Q and it's degree of extension.

[5]

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- b) Suppose α is a rational root of a monic polynomial in $\mathbb{Z}[x]$ then prove that α is an integer. [3]
- c) Determine whether the polynomial $p(x) = (x-2)^2 \in \mathbb{Q}[x]$ is separable over \mathbb{Q} . [2]
- Q4)** a) Let $\phi: F \rightarrow G$ be an isomorphism of fields. Let $f(x) \in F[x]$ and $g(x) \in G[x]$ be the polynomial obtained by applying ϕ to the coefficients of $f(x)$. Let E_1 be a splitting field for $f(x)$ over F and E_2 be splitting field for $g(x)$ over G , then prove that the isomorphism ϕ extends to an isomorphism. $\sigma: E_1 \rightarrow E_2$ [5]
- b) Define algebraic closure of a field. If K is an algebraically closed field and F is a subfield of K then prove that the collection of elements of K that are algebraic over F is an algebraic closure of F . [3]
- c) Show that doubling the cube is impossible by using straightedge and compass. [2]
- Q5)** a) Let E be the splitting field over F of the polynomial $f(x) \in F[x]$ then prove that $|\text{Aut}(K/F)| \leq [E:F]$. [5]
- b) Find all automorphisms of $\mathbb{Q}(\sqrt{2})$ over \mathbb{Q} . Is the extension $\mathbb{Q}(\sqrt{2})$ of \mathbb{Q} Galois? [3]
- c) Prove that $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$ are not isomorphic. [2]
- Q6)** a) Show that the Galois group of $x^3 - 2 \in \mathbb{Q}[x]$ is the symmetric group on three letters. [5]
- b) Show that any quadratic extension K of any field F of characteristic not equal to two is Galois. [3]
- c) Find the discriminant D of a polynomial $f(x) = x^3 - x + 1$ in $\mathbb{Q}[x]$. [2]

- Q7)** a) State the fundamental theorem of Galois theory. [5]
- b) Prove that any cyclic extension of degree n over a field F of characteristic not dividing n which contains the n th root of unity is of the form $F(\sqrt[n]{a})$ for some $a \in F$. [5]
- Q8)** a) Show that the field F_p^n is the splitting field of $x^{p^n} - x$ over F_p with cyclic Galois group of order ' n ' generated by the Frobenius automorphisms σ_p . Hence show that the subfield of F_p^n are all Galois over F_p . [5]
- b) Show that the field generated over F by α and β is the field generated by β over the field $F(\alpha)$ generated by α . [5]

