Total No. of Questions :8]

SEAT No. :

P1392

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M.Sc. / M.A.

MATHEMATICS

MT - 602 : Differential Geometry

(2008 Pattern) (Semester - II)

Time : 3 Hours] *[Max. Marks :80*

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to the right indicate full marks.*

Q1) a) Let S be an n-surface in $\mathbb{R}^{n+1}$, let $\alpha : I \to S$ be a parametrized curve in S, let $t_0 \in I$ and $V \in S_{\alpha(t_0)}$. Prove that there exists a unique vector field V tangent to S along α and $V(t_0) = V$. **[6]**

b) Find the velocity, acceleration and speed of the curve $\alpha(t) = (cost, sint)$. **[4]**

c) Consider a vector field $X(x_1, x_2) = (x_1, x_2, 1, 0)$ on $\mathbb{R}^2$. For $t \in \mathbb{R}$ and $p \in \mathbb{R}^2$, let $\phi_t(p) = \alpha_p(t)$ where α is the maximal integral curve of X through P. Show that $F(t) = \phi_t$ is a homomorphism of additive group of real numbers into the invertible linear maps of the plane. **[6]**

Q2) a) Let S be a connected n - surface in $\mathbb{R}^{n+1}$. Show that on S there exists exactly two smooth unit normal vector fields N_1 and N_2. **[6]**

b) Show that the Weingarten map of the n-sphere of radius r oriented by inward normal is multiplication by $\frac{1}{r}$. **[5]**

c) Show that the 1-form n on $\mathbb{R}^2$-{0} defined by

$$n = \frac{-x_2}{x_1^2 + x_2^2} dx_1 + \frac{x_1}{x_1^2 + x_2^2} dx_2 \text{ is not exact.}$$ **[5]**

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Q3) a) Show that the covariant differentiation has the following property:

$(X.Y)' = X'.Y + X.Y'$ **[5]**

b) Let $a,b,c,d \in \mathbb{R}$ be such that $ac\text{-}b^2 > 0$. Show that the maximum and minimum values of the function $g(x_1, x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2$ on the unit circle $x_1^2+x_2^2 = 1$ are eigen values of the matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$. **[6]**

c) If an n-surface S contains a straight line segment, then show that it is geodesic in S. **[5]**

Q4) a) Let U be an open subset of $\mathbb{R}^{n+1}$ and $f : U \to \mathbb{R}$ be a smooth function. Let $S=f^{-1}(c)$ $c \in \mathbb{R}$ and $\nabla f(q) \neq 0\ \forall q \in S$. If $g : U \to \mathbb{R}$ is smooth function and $P \in S$ is an extreme point of g on S, then show that $\exists$ a real number λ such that $\nabla g(p) = \lambda \nabla f(p)$. **[6]**

b) Let S denote a cylinder $x_1^2 + x_2^2 = r^2$ of radius r in $\mathbb{R}^3$. Show that α is a geodesic of S if and only if α is of the form $\alpha(t) = (r\cos(at + b), r\sin(at + b), ct + d)$ for some real numbers a,b,c,d. **[6]**

c) Sketch the following vector fields on $\mathbb{R}^2$: $X(P) = (P, X(P))$ where

i) $X(p) = -p$

ii) $X(x_1, x_2) = (-x_2, x_1)$. **[4]**

Q5) a) Show that the Weingarten map Lp is self-adjoint (that is

$L_p(v).w = v.L_p(w)\ \forall v, w \in S_p$.) **[6]**

b) Find the integral curve of the vector field X given by

$X(x_1,x_2) = (x_1,x_2\text{-}x_2,x_1)$ through the point (1, 0). **[5]**

c) Let $\alpha(t) = (x(t), y(t))$ be a local parametrization of the oriented plane curve C. Show that $K o \alpha = \dfrac{x'y'' - y'x''}{(x'^2 + y'^2)^{3/2}}$. **[5]**

Q6) a) Prove that on each compact oriented n-surface S in $\mathbb{R}^{n+1}$ there exists a point p such that the second fundamental form at p is definite. **[6]**

b) Show that speed of geodesic is constant. **[5]**

c) Show that the graph of any smooth function $f:\mathbb{R}^n \to \mathbb{R}$ is an n-surface in $\mathbb{R}^{n+1}$ **[5]**

Q7) a) Let U be an open subset of $\mathbb{R}^{n+1}$ and $f:\mathrm{U} \to \mathbb{R}$ be a smooth function. Let $p \in \mathrm{U}$ be a regular point of f and let $c=f(p)$. Show that the set of all vectors tangent to $f^{-1}(c)$ at p is equal to $[\nabla f(p)]^{\perp}$. **[6]**

b) Let C be connected oriented plane curve and $\beta:\mathrm{I} \to \mathrm{C}$ be a unit speed global parametrization of C. Show that β is either one to one or periodic. **[5]**

c) Find the curvature of the circle with center (a, b) and radius r oriented by outward normal. **[5]**

Q8) a) Let S be an n-surface in $\mathbb{R}^{n+1}$ and $p \in \mathrm{S}$. Prove that there exists an open set V about p in $\mathbb{R}^{n+1}$ and a parametrized n-surface $\phi:\mathrm{U} \to \mathbb{R}^{n+1}$ such that ϕ is one to one map from U onto $\mathrm{V} \cap \mathrm{S}$. **[8]**

b) Let S be the ellipsoid $\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}+\frac{x_3^2}{c^2}=1$ a,b,c all non-zero, oriented by the outward normal. Show that the Gaussian curvature of S is

$$\mathrm{K}(p) = \frac{1}{a^2b^2c^2\left(\frac{x_1^2}{a^4}+\frac{x_2^2}{b^4}+\frac{x_3^2}{c^4}\right)}.$$ **[8]**

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