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P1389

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M.A./M.Sc.

MATHEMATICS

MT-504 : Number Theory

(2008 Pattern) (Semester - I)

Time : 3 Hours] *[Max. Marks :80*

Instructions to the candidates:

1) ***Attempt any five questions.***
2) ***Figures to the right indicate full marks.***

Q1) a) Let p denote a prime. Then prove that $x^2 \equiv -1 (\text{mod } p)$ has solutions if and only if $p = 2$ or $p \equiv 1 (\text{mod } 4)$. **[6]**

b) Show that the product of three consecutive integers is divisible by 504 if the middle one is cube. **[5]**

c) Determine the value of $999^{179} (\text{mod } 1763)$. **[5]**

Q2) a) If $f(n) = \sum_{d|n} \mu(d) F(n/d)$ for every positive integer n then prove that $F(n) = \sum_{d|n} f(d)$. **[6]**

b) Find all primes p such that $x^2 \equiv 13 (\text{mod p})$ has a solution. **[5]**

c) Let x & y be any two real numbers then prove that **[5]**

i) $[x]+[y] \leq [x+y] \leq [x]+[y]+1$

ii) $[x]+[-x] = \begin{cases} 0 & \text{if } x \text{ is an integer} \\ -1 & \text{otherwise.} \end{cases}$

P.T.O.

Q3) a) If p and q are distinct odd primes then prove that $\left(\frac{p}{q}\right)\left(\frac{q}{p}\right)=(-1)^{\{(p-1)/2\}\{(q-1)/2\}}$. **[6]**

b) For every positive integer n, prove that $\delta(n)=\prod_{P^{\alpha}\|n}\left(\frac{P^{\alpha+1}-1}{P-1}\right)$. **[5]**

c) Prove that if a prime p is a quadratic residue of an odd prime q and p is of the form $4k + 1$ then q is a quadratic residue of p. **[5]**

Q4) a) Show that : **[8]**

i) The norm of product equals the product of norms $N(\alpha\beta) = N(\alpha).N(\beta)$.

ii) $N(\alpha) = 0$ if and only if $\alpha = 0$.

iii) The norm of an integer in $Q(\sqrt{m})$ is a rational integer.

b) What is the highest power of 2 dividing 533!. **[4]**

c) Find the minimal polynomial of $1+\sqrt{2}+\sqrt{3}$. **[4]**

Q5) a) Prove that every Euclidean quadratic field has the unique factorization property. **[8]**

b) If α is an algebraic integer then prove that there exists an integer b such that $b\alpha$ is an algebraic integer. **[4]**

c) Show that there is no x for which both $x\equiv 29 \pmod{52}$ and $x\equiv 19 \pmod{72}$. **[4]**

Q6) a) State and prove the Chinese remainder theorem. **[6]**

b) Find all integers x & y that satisfy $147x + 258y = 369$. **[5]**

c) Evaluate : $\left(\frac{-23}{83}\right)$. **[5]**

Q7) a) Let $Q(\sqrt{m})$ have the unique factorization property then prove that any rational prime p is either a prime π of the field or a product $\pi_1\pi_2$ of two primes, not necessarily distinct of $Q(\sqrt{m})$. **[6]**

b) Prove that $\sum_{j=1}^{p-1}\left(\frac{j}{p}\right)=0$, P an odd prime. **[5]**

c) What is the last digit in the ordinary decimal representation of 2^{400}? **[5]**

Q8) a) Let a, b and c be integer with not both a and b equal to 0 and let $g = gcd(a, b)$ and $ax + by = c$.

i) If $g \nmid c$ then show that the equation $ax + by = c$ has no solution in integer.

ii) If $g|c$ then prove that $ax + by = c$ has infinitely many solutions. **[6]**

b) Show that $61!+1\equiv 63!+1\equiv 0 \pmod{71}$. **[5]**

c) Prove that $\prod_{d|n} d = n^{d(n)/2}$. **[5]**

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