Total No. of Questions : 8]

SEAT No. :

P1402

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[5221]-41
M.A./M.Sc.
MATHEMATICS
MT - 801:Field Theory
(2008 Pattern) (Semester - IV) (Old)

Time : 3 Hours] *[Max. Marks : 80*

Instructions to the candidates:

1) ***Attempt any five questions.***
2) ***All questions carry equal marks.***
3) ***Figures to the right indicate full marks.***

Q1) a) Let $f(x) \in \mathbb{Z}[x]$ be a primitive polynomial. Prove that $f(x)$ is reducible over $\mathbb{Q}$ if and only if $f(x)$ is reducible over $\mathbb{Z}$. **[6]**

b) Show that the polynomial $x^7 - 10x^5 + 15x + 5$ is not solvable by radicals over $\mathbb{Q}$. **[6]**

c) Show that there exists an angle that cannot be trisected by using ruler and compass only. **[4]**

Q2) a) Let $f(x) \in \mathbb{Q}[x]$ be a monic irreducible polynomial over $\mathbb{Q}$ of degree *p*, *p* is prime. If $f(x)$ has exactly two non real roots in $\mathbb{C}$, then show that the Galois group of $f(x)$ is isomorphic to S_p, where S_p is a symmetric group on a set with *p* symbols. **[8]**

b) Show that if a real number $x > 0$ is constructible, then $\sqrt{x}$ is also constructible. **[4]**

c) Show that $1 + x + \ldots + x^{p-1} \in \mathbb{Q}[x]$ is irreducible over $\mathbb{Q}$, where *p* is prime. **[4]**

Q3) a) Let $f(x) = a_0 + a_1x + \ldots + a_nx^n \in \mathbb{Z}[x], n \geq 1$. If there is a prime *p* such that $p^2 \nmid a_0, p \mid a_1, p \mid a_2, \ldots, p \mid a_{n-1}, p \nmid a_n$, then show that $f(x)$ is irreducible over $\mathbb{Q}$. **[6]**

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b) Show that the Galois group $G(Q(\alpha)/Q)$, where $\alpha^5=1$ and $\alpha\neq 1$, is isomorphic to the cyclic group of order 4. **[6]**

c) Find the smallest extension of $\mathbb{Q}$ having a root of $x^4-2\in\mathbb{Q}[x]$. **[4]**

Q4) a) Show that if K is a field of characteristic $p\neq 0$, then K is perfect if and only if $K^p=K$, *i.e.*, if and only if every element of K has p^{th} root in K. **[8]**

b) Let F be a field and $p(x)$ be an irreducible polynomial in $F[x]$. Then show that there exists field extension E of F in which $p(x)$ has a root. **[4]**

c) Let E be an extension field of F. If $a\in E$ has a minimal polynomial of odd degree over F, show that $F(a)=F(a^2)$. **[4]**

Q5) a) Let E be an extension of field F, and let $u\in E$ be algebraic over F. Let $p(x)\in F(x]$ be a polynomial of the least degree such that $p(u)=0$. Prove that **[6]**

i) $p(x)$ is irreducible over F.

ii) If $g(x)\in F[x]$ is such that $g(u)=0$, then $p(x)|g(x)$.

b) Show that the degree of extension of the splitting field of $x^3-2\in\mathbb{Q}[x)$ is 6. **[6]**

c) Prove that in a finite field, every element can be written as the sum of two squares. **[4]**

Q6) a) Let E and E' be algebraic closures of a field F. Show that E is isomorphic with E' under an isomorphism that is an identity on F. **[6]**

b) Let E be the splitting field of $x^n - a \in F[x]$. Show that $G(E/F)$, the Galois group, is solvable. **[6]**

c) Let E be the splitting field of a polynomial of degree n over a field F. Show that $[E:F] \leq n!$. **[4]**

Q7) a) Prove that any irreducible polynomial $f(x)$ over a field of characteristic 0 has simple roots. Also show that any irreducible polynomial $f(x)$ over a field F of characteristic $p \neq 0$ has multiple roots and only if there exists $g(x) \in F(x]$ such that $f(x) = g(x^p)$. **[8]**

b) Define simple extension. Show that every finite separable extension of a field F is a simple extension. **[4]**

c) Show that the field generated by a root of $x^3 - x - 1$ over $\mathbb{Q}$ is not normal over $\mathbb{Q}$. **[4]**

Q8) a) Show that every finite field F with p^n elements is the splitting field of $x^{p^n} - x \in F_p[x]$, where F_p is the subfield of F with p elements. Also show that any two finite fields with p^n elements are isomorphic. **[8]**

b) Show that a finite field F of p^n elements has exactly one subfield with p^m elements for each divisor m of n. **[4]**

c) Show that if $f(x) \in F[x]$ is irreducible over F, then all roots of $f(x)$ have the same multiplicity. **[4]**

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