Total No. of Questions : 8]

SEAT No. :

P1394

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[5221]-24
M.A./M.Sc.
MATHEMATICS
MT-604 : Complex Analysis
(2008 Pattern) (Semester-II)

Time : 3 Hours] *[Max. Marks : 80*

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to the right indicate full marks.*

Q1) a) Prove that the radii of convergence of $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=1}^{\infty} n a_n z^{n-1}$ are same. **[6]**

b) Let G be an open connected subset of $\mathbb{C}$ and let $f : \mathrm{G} \to \mathbb{C}$ be differential such that $f'(z) = 0$ for all z in G. Prove that f is constant. **[5]**

c) Find the radius of convergence of each of the series $\sum_{n=1}^{\infty} 2^n z^n$ and $\sum_{n=1}^{\infty} z^{n!}$. **[5]**

Q2) a) Let G be an open disc in $\mathbb{C}$ and let $u : \mathrm{G} \to \mathbb{R}$ be a harmonic function. Prove that u has a harmonic conjugate. **[6]**

b) Show that a real-valued analytic function is constant. **[5]**

c) Show that a Mobius transformation is a combination of translations, dilations and the inversion. **[5]**

Q3) a) Let z_1, z_2, z_3, z_4 be four distinct points in $\mathbb{C}_\infty$. Prove that the cross ratio (z_1, z_2, z_3, z_4) is a real number if and only if all four points lie on a circle. **[6]**

b) Let G be a region and suppose $f : \mathrm{G} \to \mathbb{C}$ is analytic such that $f(\mathrm{G})$ is contained in a subset of a circle. Show that f is constant. **[5]**

c) Prove that a Mobius transformation has ∞ as its only fixed point if and only if it is a translation. **[5]**

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Q4) a) Let $f : \mathrm{G} \to \mathbb{C}$ be analytic and suppose the closure of the disc B(a; r) is a subset of G. Prove that if $\gamma(t) = a + re^{it}, 0 \le t \le 2\pi$, then **[6]**

$$f(z) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(w)}{w-z}dw \text{ for } |z-a| < r.$$

b) Let f be analytic in the disc B(a; R) and suppose that γ is a closed rectifiable curve in B(a; r). Prove that $\int_{\gamma} f = 0$. **[5]**

c) Evaluate the integrals: **[5]**

i) $\int_{|z|=1} \frac{e^{iz}}{z^2}dz$ ii) $\int_{|z|=2} \frac{1}{z^2+1}dz$

Q5) a) State and prove Liouville's theorem. Hence deduce that sin z is not bounded. **[6]**

b) State and prove the maximum modulus theorem. **[5]**

c) Let f and g be two entire functions such that $fg \equiv 0$. Prove that $f \equiv 0$ or $g \equiv 0$. **[5]**

Q6) a) Let G be a simply connected region and $f : \mathrm{G} \to \mathbb{C}$ be analytic in G. Prove that f has a primitive. **[6]**

b) Let $\gamma(\theta) = \theta e^{i\theta}$ for $0 \le \theta \le 2\pi$ and $\gamma(\theta) = 4\pi - \theta$ for $2\pi \le \theta \le 4\pi$.

Evaluate $\int_{\gamma} \frac{dz}{z^2+\pi^2}$. **[5]**

c) Show that if $f : \mathrm{G} \to \mathbb{C}$ is analytic and one-one, then $f'(z) \ne 0$ for any $z \in \mathrm{G}$. **[5]**

Q7) a) State and prove Goursat's theorem. **[6]**

b) Let G be a bounded region and suppose f is continuous on $\overline{G}$ and analytic on G. Show that if $|f(z)| = 2$ for all z on the boundary of G, then either f is a constant function or f has a zero in G. **[5]**

c) State and prove Schwarz's lemma. **[5]**

Q8) a) Suppose f has an essential singularity at $z = a$. Prove that for every $\delta > 0$, the set $f\,[ann(a;\,0,\,\delta)]$ is dense in $\mathbb{C}$. **[6]**

b) Classify the singularities of $f(z) = \dfrac{z^2 - 1}{z(z-1)^2}$. Also, find the residue at each singularity. **[5]**

c) Using residue calculus, show that $\int_0^{\pi} \dfrac{d\theta}{a + \cos\theta} = \dfrac{\pi}{\sqrt{a^2 - 1}}$. **[5]**

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